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On the Approximation of Kernel Operators by Operators of Finite Rank

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1. INTRODUCTION

In this paper we are concerned with linear operators from a space $L^p(\mu)$ into a space $L^q(\nu)$, representable by measurable kernels and satisfying certain special conditions originally defined and discussed by Hille and Tamarkin [5]. These operators can be viewed as generalizations of the Hilbert–Schmidt operators; they are known to form Banach spaces under a certain natural norm that coincides with the Hilbert–Schmidt norm if $p = q = 2$. Since a Hilbert–Schmidt operator can be approximated in the Hilbert–Schmidt norm by operators of finite rank, it is natural to ask if this situation extends to the case where the exponents p, q are distinct from 2. Obviously, certain extreme cases have to be excluded here: For example, if $p = 1$ and $q = \infty$, then any continuous linear map $L^p(\mu) \to L^q(\nu)$ has a kernel of the type we are considering, but such a map need not even be weakly compact. On the other hand, it is well known that the operators in question are compact for $1 < p, q < \infty$ (see, e.g., Luxemburg and Zaanen [7]), and results in the direction indicated have in fact been obtained under additional assumptions on p and q (Jörgens [6, Satz 11.6]). It is our aim to remove these assumptions, and to identify the Hille–Tamarkin operators from $L^p(\mu)$ into $L^q(\nu)$ with the elements of a completed normed tensor product of $L^{p'}(\mu)$ and $L^q(\nu)$ as defined in [9]. This implies, in particular, that (for $1 < p, q < \infty$) the space of Hille–Tamarkin operators $L^p(\mu) \to L^q(\nu)$ is a reflexive Banach lattice, with dual given by the Hille–Tamarkin operators $L^{p'}(\mu) \to L^{q'}(\nu)$, in complete analogy to the case $p = q = 2$.

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In the following, E will always denote the space $L^p(\mu)$ $(1 \leqslant p \leqslant \infty)$ constructed over a σ-finite measure space (X, Σ, μ). Likewise, F denotes the space $L^q(\nu)$ $(1 \leqslant q \leqslant \infty)$ for a σ-finite measure space (Y, Ω, ν). We denote by p', q' the conjugate exponents as usual. $\mathscr{L}^r(E, F)$ is the space of all order-bounded linear maps from E into F, which is a Banach lattice for the natural order and the r-norm $T \to \|T\|_r = \||T|\|$ (see, e.g., [10]). An operator $T \in \mathscr{L}^r(E, F)$ is called a *kernel operator* if there exists a $\mu \otimes \nu$-measurable function $K(s, t)$ on $X \times Y$ such that for each $f \in E$, $s \to K(s, t) f(s)$ is μ-integrable for ν-almost every $t \in Y$, and $Tf(t) = \int K(s, t) f(s)\, d\mu(s)$ holds almost everywhere (ν). We denote by $\mathscr{A}_{pq}$ the vector space of all kernel operators $E \to F$. If $T \in \mathscr{A}_{pq}$ has the kernel $K(s, t)$, then the function $|K(s, t)|$ is a kernel for the modulus $|T|$ (see [8] or [10]). Consequently, $\mathscr{A}_{pq}$ is a Banach sublattice of $\mathscr{L}^r(E, F)$. An operator $T \in \mathscr{A}_{pq}$ is called a *Hille–Tamarkin* operator if

$$k(t) = \left(\int |K(s, t)|^{p'} d\mu(s)\right)^{1/p'}, \tag{1}$$

(respectively, $k(t) = \sup \operatorname{ess}_s |K(s, t)|$ if $p = 1$) is finite almost everywhere (ν) and defines a function $k \in F$. $\mathscr{H}_{pq}$ denotes the space of all Hille–Tamarkin operators from E into F. $\mathscr{H}_{pq}$ is obviously a sublattice of $\mathscr{L}^r(E, F)$, and the function $T \mapsto \|k(t)\|_q$ is a lattice norm on $\mathscr{H}_{pq}$. It is immediately clear that the kernel $K(s, t)$ of an operator $T \in \mathscr{H}_{pq}$ defines a weakly ν-measurable map $t \mapsto g_t$, where $g_t(s) = K(s, t)$, from Y into $L^{p'}(\mu)$. If $1 < p, q < \infty$, we will show that this map is always ν-measurable, and actually contained in $L^q_H(\nu)$, where $H = L^{p'}(\mu)$.

2. Vector Valued L^p-Spaces

In this section, G and H are arbitrary Banach lattices. As usual, $L^p_H(\mu)$ denotes the space of equivalence classes of μ-measurable functions $g: X \to H$ with $s \mapsto \|g(s)\| \in L^p(\mu)$. We want to identify $L^p_H(\mu)$ with a completed tensor product of $L^p(\mu)$ and H, and to determine the dual spaces $(L^p_H(\mu))'$. For this we need some definitions from [9]. In order to simplify the presentation, we will assume that H is the image of its second dual under a positive projection of norm 1 (see [9]).

A linear map T from G into H is called *majorizing* if for any null sequence $\{x_n\}$ in G the sequence $\{Tx_n\}$ is order bounded in H. If T is majorizing, then the image under T of the unit ball U_G of G is order-bounded in H, and the map $T \mapsto \|T\|_m = \|\sup\{Tx: x \in U_G\}\|$ is a norm on the linear space $\mathscr{L}^m(G, H)$ of all majorizing maps $G \to H$. Under this norm and the natural order, $\mathscr{L}^m(G, H)$ is a Banach lattice and an ideal of $\mathscr{L}^r(G, H)$. Dually, we call an operator $T \in \mathscr{L}(G, H)$ *cone absolutely summing* (c.a.s.) if T maps positive, summable sequences in G into absolutely summable sequences in H. The mapping $T \mapsto \| T \|_l = \sup\{\Sigma \| Tx_n \|: 0 \leq x_n \in G, \| \Sigma x_n \| \leq 1\}$ is a norm on the linear space $\mathscr{L}^l(G, H)$ of all c.a.s. maps $G \to H$ and under this norm and the natural order, $\mathscr{L}^l(G, H)$ is a Banach lattice and an ideal in $\mathscr{L}^r(G, H)$.

The preceding two classes of linear maps are dual to each other in the following sense: $T \in \mathscr{L}(G, H)$ is majorizing (c.a.s.) if and only if the adjoint $T' \in \mathscr{L}(H', G')$ is c.a.s. (majorizing), and then $\| T \|_m = \| T' \|_l$ (respectively, $\| T \|_l = \| T' \|_m$). Both $\mathscr{L}^m(G, H)$ and $\mathscr{L}^l(G, H)$ contain the operators of finite rank, and we denote by $G \tilde{\otimes}_m H$ the closure of $G \otimes H$ in $\mathscr{L}^m(G', H)$, while $G \tilde{\otimes}_l H$ denotes the closure of $G \otimes H$ in $\mathscr{L}^l(G', H)$. It turns out [9] that $G \tilde{\otimes}_m H$ and $G \tilde{\otimes}_l H$ are Banach sublattices of $\mathscr{L}^m(G', H)$ and $\mathscr{L}^l(G', H)$, respectively, and that $G \tilde{\otimes}_m H$ is isomorphic to $H \tilde{\otimes}_l G$ via the extension of the transposition map $x \otimes y \mapsto y \otimes x$. If G is a space $L^1(\mu)$, then the l-norm on $G \otimes H$ coincides with the π-norm (greatest crossnorm), hence $G \tilde{\otimes}_l H = G \tilde{\otimes}_\pi H$. Correspondingly, $G \tilde{\otimes}_m H = G \tilde{\otimes}_\pi H$ if H is a space $L^1(\nu)$. On the other hand, $G \tilde{\otimes}_l H = G \tilde{\otimes}_\epsilon H$ if G is a space $C(Z)$ (continuous functions on the compact space Z with the sup-norm) and $G \tilde{\otimes}_m H = G \tilde{\otimes}_\epsilon H$ if H is a space $C(Z)$, ϵ denoting the least crossnorm. We will need the following basic properties of the tensor products just defined:

PROPOSITION 1. *The dual of $G \tilde{\otimes}_m H$ is canonically isomorphic (as a Banach lattice) to $\mathscr{L}^m(G, H')$, and $(G \tilde{\otimes}_l H)'$ is isomorphic to $\mathscr{L}^l(G, H')$. If G is a space $L^p(\mu)$ $(1 \leq p < \infty)$, then $G \tilde{\otimes}_l H$ can be canonically identified with the Banach lattice $L^p_H(\mu)$.*

A proof can be found in [9], except for the last statement which was proved in [2]. The duality relations expressed in Proposition 1 can be further refined if G is a space $L^p(\mu)$ and if H is reflexive. We need the following Lemma which is due to Grothendieck [3]; a short proof can be found in [10].

LEMMA. *Let J, K be Banach spaces, and let T be an integral linear map (in the sense of Grothendieck [3]) from J into K. If K is a separable dual or if K is reflexive, then T is nuclear.*

THEOREM 1. *Let (X, Σ, μ) be a σ-finite measure space, and suppose that H is a separable dual, or that H is reflexive. Then $\mathscr{L}^l(L^p(\mu), H)$ is isomorphic as a Banach lattice to $L^{p'}_H(\mu)$ under the correspondence $T \mapsto g$ given by the identity of bilinear forms*

$$\langle Tf, y' \rangle = \int f(s) \langle y', g(s) \rangle \, d\mu(s) \tag{2}$$

on $L^p(\mu) \times H'$.

Proof. The general case can be reduced to the case of a finite measure space by a standard procedure, so let us assume that $\mu(X) < \infty$. Let $T \in \mathscr{L}^l(L^p(\mu), H)$, and denote by T_0 its restriction to $L^\infty(\mu)$. T_0 is integral [9], hence nuclear by the Lemma, and since $T_0'(H')$ is contained in the band $L^1(\mu)$ of $L^\infty(\mu)'$, we conclude that $T_0 \in L^1(\mu) \tilde{\otimes}_\pi H$. Thus T_0 satisfies (2) for a unique $g \in L^1 \tilde{\otimes}_\pi H = L_H{}^1(\mu)$ and this representation extends to all of $L^p(\mu)$ by a continuity argument. Since g is automatically μ-measurable, it remains to show that the function $s \to \| g(s) \|$ is contained in $L^{p'}(\mu)$. For this, we first note that $g(s)$ is contained in a separable subspace of H for μ-almost all $s \in X$. In fact, since T_0 is nuclear, there exists a closed separable subspace H_0 of H containing $T_0(L^\infty(\mu))$. But T_0 is nuclear as a map into H_0 as well (we put $H = H_0$ if H is separable to begin with), and the uniqueness of the function g then yields the desired conclusion. Hence there exists a sequence $\{y_n'\}$ contained in the unit ball of H' such that $\| y \| = \sup_n \langle y, y_n' \rangle$ holds for all $y \in H_0$. Now $\langle g(s), y_n' \rangle = (T'y_n')(s)$ for μ-almost every $s \in X$ and, since T' is majorizing, there exists a function $h \in L^{p'}$ such that $T'y_n'$ is contained in the interval $[-h, h]$ of $L^{p'}$ for all $n \in \mathbb{N}$. Consequently, there exists a μ-null set N in X such that $\langle g(s), y_n' \rangle \leq | h(s) |$ for all s in the complement of N and for all n, hence $g \in L_H^{p'}(\mu)$. Finally, it is not hard to verify that the correspondence $T \to g$ just established defines an isomorphism of Banach lattices from $\mathscr{L}^l(L^p(\mu), H)$ onto $L_H^{p'}(\mu)$.

Remark. Of course, Theorem 1 remains true if H is only supposed to be a Banach space, except that $\mathscr{L}^l(L^p, H)$ and $L_H^{p'}$ do not carry a lattice structure in this case. For $p = 1$ and H a separable dual, Theorem 1 is the classical Dunford–Pettis Theorem (note that $\mathscr{L}^l(L^1(\mu), H) = \mathscr{L}(L^1(\mu), H)$). For $1 < p < \infty$ and H separable and reflexive, the result (more precisely the coincidence of $(L_{H'}^p)'$ and $L_H^{p'}$) can be found, e.g., in Bourbaki [1, Section 2, Exercise 21]. Chaney [2] showed that no separability assumptions are needed if H is reflexive.

The following is now an immediate consequence of Theorem 1 and Proposition 1.

COROLLARY. *Let $E = L^p(\mu)$, $F = L^q(\nu)$ for σ-finite measures μ and ν, and let $1 < p, q < \infty$. Then $E \tilde{\otimes}_m F$ and $E \tilde{\otimes}_l F$ are reflexive Banach lattices with duals $E' \tilde{\otimes}_m F'$ and $E' \tilde{\otimes}_l F'$, respectively. In particular, $E' \tilde{\otimes}_m F = \mathscr{L}^m(E, F) = L_{E'}^q(\nu)$ and $E' \tilde{\otimes}_l F = \mathscr{L}^l(E, F) = L_F^{p'}(\mu)$.*

3. Applications

The connection between the operators in $\mathscr{L}^m(E, F)$ and the Hille–Tamarkin operators now becomes transparent: If $1 < p, q < \infty$, if $T \in \mathscr{L}^m(E, F)$, and if $K(s, t)$ is the kernel of T, then T' is contained in $\mathscr{L}^l(F', E')$ with kernel

$K(t, s)$, and the function $g\colon Y \to E'$ associated with T by Theorem 1 is given by

$$g(t)(s) = K(s, t),$$

hence $T \in \mathscr{H}_{pq}$. Conversely, if $T \in \mathscr{H}_{pq}$, then the function k of (1) is contained in L^q, hence $T \in \mathscr{L}^m(E, F)$ by definition. Moreover, it is clear that $\mathscr{H}_{pq}$ thus becomes a normed sublattice of $\mathscr{L}^m(E, F)$, and the same is true for the remaining cases $p, q = 1$ or ∞. Thus $\mathscr{H}_{pq}$, with its natural norm and order, is the normed sublattice of $\mathscr{L}^m(E, F)$ consisting of the majorizing kernel operators. We define $\mathscr{J}_{pq}$ to be the space $\mathscr{L}^l(E, F) \cap \mathscr{A}_{pq}$ with the induced l-norm. Since $\mathscr{A}_{pq}$ is a Banach space for the r-norm and since both the m-norm and the l-norm are greater than the r-norm, it is clear that $\mathscr{H}_{pq}$ and $\mathscr{J}_{pq}$ are Banach lattices ($1 \leqslant p, q \leqslant \infty$). It follows from the definition of these spaces that $T \in \mathscr{H}_{pq}$ if and only if $T' \in \mathscr{J}_{q'p'}$, and that the respective norms of T and T' coincide. For the sake of completeness, we write down the explicit formulae for these norms, identifying an operator T with its kernel K: For $1 < p, q < \infty$,

$$\| K \|_m = \left[\int \left(\int | K(s, t)|^{p'} \, d\mu(s) \right)^{q/p'} d\nu(t) \right]^{1/q} \tag{3}$$

is the norm of $K \in \mathscr{H}_{pq}$, while for $K \in \mathscr{J}_{pq}$ the norm of K is given by

$$\| K \|_l = \left[\int \left(\int | K(s, t)|^{q} \, d\nu(t) \right)^{p'/q} d\mu(s) \right]^{1/p'}, \tag{4}$$

corresponding formulae holding if $p = 1$ and/or $q = \infty$. We summarize:

Proposition 2. *The spaces $\mathscr{H}_{pq}$ of majorizing and $\mathscr{J}_{pq}$ of c.a.s. kernel operators $L^p(\mu) \to L^q(\nu)$ are Banach lattices; for each pair (p, q), $\mathscr{H}_{pq}$ is isomorphic to $\mathscr{J}_{q'p'}$ by transposition of kernels. If $q = p' < \infty$, then $\mathscr{H}_{pp'} = \mathscr{J}_{pp'} = L^{p'}(\mu \otimes \nu)$.*

For $p = q = 2$, the space of Hilbert–Schmidt operators emerges as a special case. We note in passing that for $1 < p < \infty$ and $1 \leqslant q < \infty$ the compactness of the Hille–Tamarkin operators $L^p(\mu) \to L^q(\nu)$ is now an easy consequence of the decomposition properties of majorizing maps (see [9]) together with the fact that every AM-space has the Dunford–Pettis property [4]. Compactness of any $T \in \mathscr{J}_{pq}$ for $1 < p \leqslant \infty$ and $1 < q < \infty$ can be verified by a corresponding argument using the Dunford–Pettis property of AL-spaces. We point out that, incidentally, the Dunford–Pettis property is also the major tool in proving the Lemma preceding Theorem 1. The following is now our main result.

Theorem 2. *If $1 < p, q < \infty$ then $\mathscr{H}_{pq} = E' \tilde{\otimes}_m F$ and $\mathscr{J}_{pq} = E' \tilde{\otimes}_l F$*

are reflexive Banach lattices with duals $\mathcal{K}_{p'q'} = E \tilde{\otimes}_m F'$ *and* $\mathcal{J}_{p'q'} = E \tilde{\otimes}_l F'$, *respectively. In particular, every majorizing* (*respectively, c.a.s.*) *linear operator* $L^p(\mu) \to L^q(\nu)$ *is a compact kernel operator and can be approximated in the m-norm* (3) (*respectively, in the l-norm* (4)) *by operators of finite rank.*

The proof is readily obtained from Proposition 1 and Theorem 1 with its Corollary.

We briefly discuss some of the cases excluded in Theorem 2.

Case 1. If $1 < p < \infty$ and $q = 1$, then every operator in $\mathcal{L}^m(E, F)$ is integral (see [9]), hence nuclear by the Lemma. Thus $\mathcal{L}^m(E, F) = \mathcal{K}_{p1} = E' \tilde{\otimes}_\pi F$, with coincidence of the respective norms. Similarly, if $p = \infty$ and $1 < q < \infty$, then we have $\mathcal{L}^l(E, F) = E' \tilde{\otimes}_\pi F$; however, $\mathcal{J}_{\infty q} = L^1(\mu) \tilde{\otimes}_\pi F$ is the space of order continuous nuclear maps $L^\infty(\mu) \to F$. On the other hand, an operator in $\mathcal{K}_{\infty q}$ or $\mathcal{J}_{p1}$ $(1 < p, q < \infty)$ need not be compact, but one always has $\mathcal{K}_{\infty q} = \mathcal{A}_{\infty q}$ and $\mathcal{J}_{p1} = \mathcal{A}_{p1}$ in these cases.

Case 2. A situation of particular interest arises for $p = \infty$, $q = 1$. By Formulae (3) and (4) above we have $\mathcal{A}_{\infty 1} = \mathcal{K}_{\infty 1} = \mathcal{J}_{\infty 1} = L^1(\mu \otimes \nu) = L^1(\mu) \tilde{\otimes}_\pi F$. Hence every kernel operator $L^\infty(\mu) \to L^1(\nu)$ is nuclear.

Case 3. Finally, we consider the case $p = 1$ and $q = \infty$. Since $\mathcal{L}(L^1(\mu), L^\infty(\nu))$ is canonically isomorphic with the space of continuous bilinear forms on $L^1(\mu) \times L^1(\nu)$ and the latter is the dual of $L^1(\mu) \tilde{\otimes}_\pi L^1(\nu) = L^1(\mu \otimes \nu)$, $\mathcal{L}(L^1(\mu), L^\infty(\nu))$ is Banach lattice isomorphic to $L^\infty(\mu \otimes \nu)$, the isomorphism being given by the formula

$$\langle Tf, g \rangle = \iint K(s, t) f(s) g(t) \, d\mu(s) \, d\nu(t) \quad (f \in L^1(\mu),\ g \in L^1(\nu)).$$

Hence we have $\mathcal{K}_{1\infty} = \mathcal{J}_{1\infty} = \mathcal{A}_{1\infty} = \mathcal{L}(E, F)$. We point out that in the present circumstances it can happen that an operator $T \in \mathcal{L}(E, F)$ which is not even weakly compact, has a modulus $|T|$ of rank 1.

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